Please go to Update page and Course page to see information about class and to keep up to date. The links are in the main page on Canvas and also on http://nature-lover.net/math. You can see old notes from spring 25 etc at this website. It will help you prepare for class.

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Today: More on exponential functions, doubling time and logarithms.

Review of compound interest when compounded multiple times:

If interest is compounded more than once a year, say n times, then need to **divide the rate by n** and **multiply the time by n**.

You get $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$

Continuous compounding : when n increases to infinity, the amount increases but very slowly, and never goes above e^{rt}.

Formula: $A(t) = Pe^{rt}$.

This is also the formula, in general, when something is growing naturally, i.e, continuously, like populations of animals,, growth of plants, spread of disease.

Table calculating $A(t) = P\left(1 + \frac{r}{n}\right)^{nt}$
When P = 1000, r = 4% = 0.04, t = 1 as n varies
(In other words, how does amount after 1 year change if you calculate interest more and more frequently?)

		Calculates	
1	1040.0000	P(1+(r/n))^(nt)	
2	1040.4	n in column 1	
3	1040.535704	P = 1000, r = 0.04, t = 1.	
4	1040.60401	Only n varies	
5	1040.645141		
6	1040.672622		
7	1040.692282		
8	1040.707044		
9	1040.718535		
10	1040.727734		
11	1040.735265		
12	1040.741543	Compare with	
13	1040.746857	This column	
14	1040.751414	e^(rt) with r = 0.04, t=1	
15	1040.755364		1040.810774
100	1040.80245		1040.810774
1000	1040.809942		1040.810774
1000000	1040.810773		1040.810774

In general, a^x gives a growth function if $a > 1$. It will be decay if $a < 1$. Same is true for e^{rt} = $(e^r)^t$: (here we have instead of a, $put\ e^r$) It will grow if $e^r > 1 \Longrightarrow r > 0$ and decay if $e^r < 1 \Longrightarrow r < 0$.

Pe^{-2t} : Example of exponential function involving decay.

Problem: $r = 0.06, n = 6,$ i.e, compounding every two months.

$A(t) = P\left(1 + \frac{0.06}{6}\right)^{6t}$ = $P(1.01^{6t})$

So in 1 year (i.e, t = 1), if P = 1000, A(1) = 1000(1.01^6) = 1061.52

What happens when n = 12, 24, 360, etc.,?

Compare with 1000($e^{0.06}$) = 1061.84

If $r = 0.12, n = 12,$ i.e, compounding every month

$A(t) = P\left(1 + \frac{0.12}{12}\right)^{12t}$ = $P(1.01^{12t})$
Approaches $Pe^{0.12t}$ as n gets bigger and bigger.

Doubling time (or tripling time, quadrupling time…) same no matter what you start with.

Example: Suppose 1000e^(rt) = 2000 in t = 5 years.

Then in 10 years, the 2000 would become 4000.

Notice : In first 5 years, it grew by 1000.
In next five years, it grew by 2000.
In 15 years, it will become 8000.
In 20 years, it will become 16000.

Logarithms

If we know rate of interest, how long would it take to double?

For that we need logarithms, which are inverse of exponential functions.
So we want to find time for P to become 2P (note: Doesn't matter what P is!)
Put $Pe^{rt} = 2P$. Cancelling P,

$e^{rt} = 2 \Longrightarrow ln(e^{rt}) = ln2 \Longrightarrow rt = ln2 \Longrightarrow t = ln2/r$.

For example, because $ln2 = 0.693$ (approximately 0.7), if the rate of growth r is 7 percent (0.07), then the doubling time will be 0.7 divided by 7 or just 10 years. In general, we can find the doubling time approximately by dividing 70 by the percentage.

Simple properties and definition of logarithms

Logarithm is just the power of a number in a certain base.
For example, if the base is 2, $log_2 16 = 4$ because $2^4 = 16$.

$$a^x = y \Longrightarrow x = log_a y$$

So, if $2^5 = 32 \ then \ 5 = log_2 32$

Similarly, if $10^2 = 100 \ then \ 2 = log_{10} 100 = \log 100$

By convention, log(x) to the base 10 is written simply as log(x).

Log(x) to the base e is written as ln(x).

Remember, log(x), ln(x), are functions, just like sin(x), cos(x), e^x, etc., These are not to be cancelled like numbers.

Basic properties of logarithms

Log(AB) = Log A + Log B, Log (A/B) = Log A – Log B.
Log (A^m) = m x (Log A)

Any exponential function has a doubling time.

In the example that we saw :

Initial amount is 1000 $. The interest rate is r.
Amount after t years is $A(t) = 1000(1+r)^t$

Suppose r = 10%. Then 1+r = 1.1.
How long it takes for amount to double, i.e, grow to 2000?
Set A(t) = 2000.
Get $2000 = 1000(1.1^t)$ => $2 = 1.1^t$ => $\ln 2 = \ln(1.1^t)$
=> $\ln 2 = t \ln (1.1)$ => t = 7.273 years.

So logarithm functions (log(x) or ln(x)) is the inverse of exponential functions. Whenever we want to solve for the exponent, we use logarithm.

So about every 7 years it doubles. (You can use the same formula to find tripling time, quadrupling time, etc., Just put ln3, ln4, etc., instead of ln2)

The doubling time is the same no matter what the initial amount is!

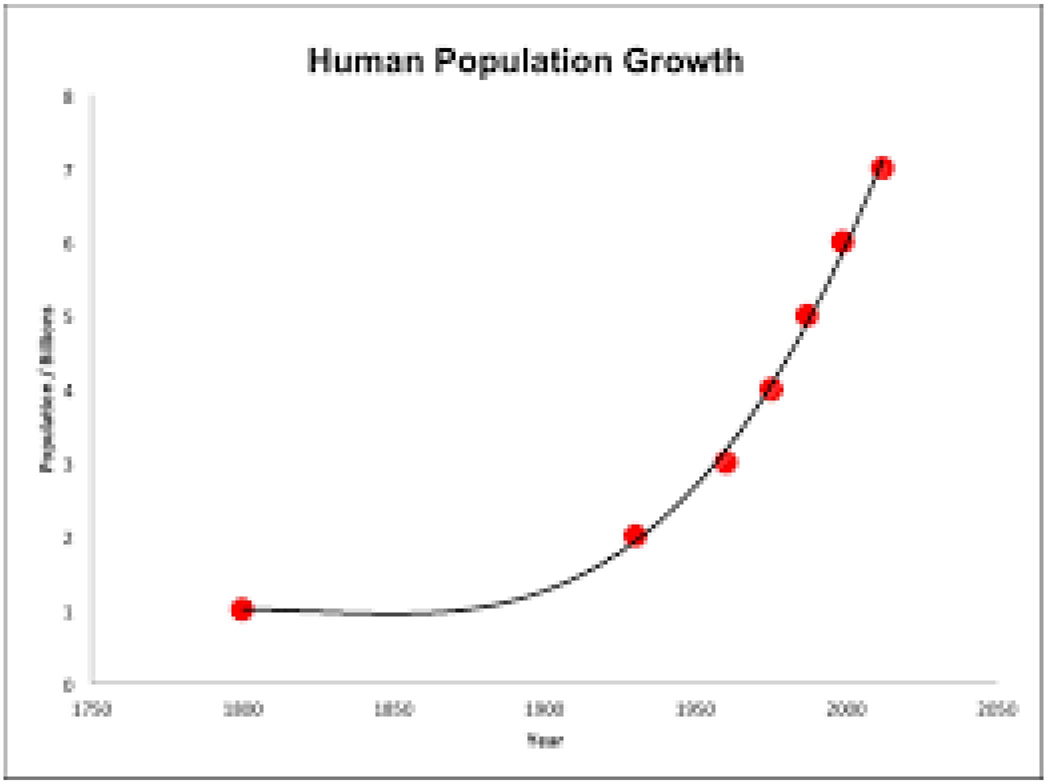
Suppose you start with a million dollars, then from the same equation get t = 7.273 as before.

Also, in first (approximately) 7 years you go from 1000 to 2000.

In the second 7 years you go from 2000 to 4000. (increases by 2000).

In the third 7 years you go from 4000 to 8000. (increases by 4000).

You can see the amount of increase in each 7 years is also a geometric sequence: 1000, 2000, 4000,… with a common ratio of 2



Notice how it is taking less and less time to double. This is not actually continuous growth because there is also death which reduces the growth, but it does involve exponential function. Took over a century to go from 1 billion to 2 billion but now we are adding a billion people every few years.

PRACTICE PROBLEMS

1. If the doubling time for a population growing exponentially is

20 years, and initial population is 1 million, what would the population be in 80 years (i.e, after it doubles 4 times) ?

2. What would population be after 400 years? Do it without repeated doubling, i.e, without going: “1million in 10 years, 2 million in 20 years,….’’

3. Find amount in an account starting with 2000 dollars with interest compounded every year at 5%, after 10 years.

4. Do problem 3 if interest is compounded 12 times a year, then do the same if interest is compounded 365 times a year.

5. Compare answers from 3 and 4 to the amount you get if interest is compounded continuously, with principal and rate remaining the same.

6. Find doubling time when interest is compounded once a year.

7. Find doubling time under continuous compounding.